A Standardized Algorithm for the Determination of Position Errors by the Example of GPS with and without 'Selective Availability'

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Abstract

Today, the accuracy of position determination ashore or at sea is no longer a technical problem, but mainly a problem of clear mathematical formulation. This paper deals with the determination of accuracy. The results of the study by Burt et al (1965), developed further by the author over the years, are taken as a basis for the discussion. This paper describes a complete algorithm which is made available in the World Wide Web. The algorithm is used to evaluate GPS measurements with and without Selective Availability and to show differences in accuracy.

Key words: *Navigation accuracy, error circle, error ellipse, dRMS, CEP, GPS*.

1 Introduction

Satellite navigation is widely used for civil applications. It can be said that shipping is the user community with the longest experience and still, if recreational vessels are included, provides the largest number of users today. Land transport, with navigation systems installed in automobiles and possibly trains, is rapidly catching up and will be by far the largest group of users in future years. Air traffic falls in between, with a heavily regulated use of equipment which has tended to slow down the application of this navigational aid. Surveying has many years of experience in satellite navigation.

In the following, the term 'satellite navigation' means the Global Positioning System 'GPS'. GPS receivers are by far the most commonly used equipment. An earlier form of satellite navigation system, known as TRANSIT, was previously used with success by both the shipping and surveying communities. The theoretical considerations made in this paper can also be applied to other systems, such as the Russian 'GLONASS' or the future European 'GALILEO' system, as well as to all other two-dimensional position-finding procedures.

The specific starting point of this investigation is the stationary use of satellite navigation. The actual questions are:

- *what magnitude of error is to be expected?*
- *for how long have position measurements to be averaged in order to obtain an error level which is smaller than a specified threshold?*

The concrete case dealt with the determination of the location of a mobile VTS system (Vessel Traffic Services system). Supported by radar, this system detects and locates shipping traffic which is then displayed on an electronic chart display and information system (ECDIS). After positioning the measuring vehicle, the geographical location of the radar antenna must first be determined since the positions of the traffic entities are calculated relative to this radar antenna. Then, radar station and shipping traffic can be displayed on the electronic chart display. Although this might seem to be a very special application, such methods are frequently used. Similar questions occur, for example, in surveying, civil construction and mining while positioning equipment.

When dealing with navigation accuracy, the form in which position errors are to be presented should first be defined. In this case, methods of error statistics as used for one-dimensional scalar quantities fail due to multi-dimensionality of the measured quantity. In practice, there exist numerous error measures which differ considerably as far as their significance is concerned (Bowditch1977, Harre 1980, Harre 1990). In the following, a calculation procedure from two-dimensional statistics, providing clear and easily understandable results, is proposed. These results, as well as the more common error quantities dRMS/2dRMS, are calculated and illustrate in the form of error contours (error ellipse/error circle) the *random* position error. Another calculation procedure is proposed to show the time-dependent convergence of the *systematic* position error due to averaging. The algorithms are implemented in MATHCAD. The relevant files can be loaded from the www (Harre 2001). They can be used directly for similar evaluations if the MATHCAD program is available.

Two data sets collected with a 12-channel GPS receiver while Selective Availability (SA) was active and after its discontinuation, were used to test the algorithms. The results also provide an indication on position accuracy improvement without SA.

2 Calculation of the random position error

2.1 Display of position measurements as 'Scatter plot'

In the case of a stationary installation of a satellite receiver and display of the measured positions in the coordinate system of a chart, one normally gets a scattering of the measured positions about a point resulting from the average values of the north and east coordinates of the individual measurements. This point is the 'Probable Position' - PP.

$$P_{PP} := \left(\bar{j}, \bar{l}\right) \qquad \textbf{(1)}$$

If the true location of the satellite receiver antenna is known (ö$_{W}$, ë$_{W}$), the individual error for the i-th position measurement is:

$$\overrightarrow{\Delta P_{w_i}} := \left(j_i - j_w, l_i - l_w\right) \qquad \textbf{(2)}$$

Since the true position of the antenna location for the application considered in this paper is not known - the procedure is actually applied to determine an estimated value for the true position, the individual errors are calculated relative to PP:

$$\overrightarrow{\Delta P_{PP_i}} := \left(j_i - j_{PP}, l_i - l_{PP}\right) \qquad \textbf{(3)}$$

This means that the entire data set must be available prior to evaluation.

The algorithm is set up so that the systematic and the random error are determined in the case of a known true position whereas only the random error relative to the averaged position is determined in the case of an unknown true position.

Subsequent to their conversion from degrees into meters (ö y; ë x) with due consideration of the meridian convergence, the individual errors are displayed in a cartesian coordinate system (Fig. 1, Fig. 2). Such a display ('Scatter plot') is informative since it does not only provide a first general idea on the scattering of the measured values, but can also show systematic error behavior, e.g. 'Random Walk' or position jumps caused by reflection.

2.2 Error circles dRMS and 2dRMS

The random errors of position measurements are then used to calculate, in the known manner, the standard deviations (ó) of the x- and y-coordinates. The first error quantity dRMS, which is easy to calculate, results from the standard deviations. This measure refers to an error radius which is calculated as follows:

$$dRMS := \sqrt{s_x^2 + s_y^2} \qquad \textbf{(4)}$$

Contrary to one-dimensional statistics, there is no fixed probability level for this error measure. The probability level (p) depends on the ratio of standard deviations (cf. Table).

ó$_y$/ó$_x$	**1dRMS**	**p(1dRMS)**	**2dRMS**	**p(2dRMS)**
0.0	1.0	0.6827	2.0	0.9545
0.25	1.0308	0.6815	2.0616	0.9591
0.5	1.1180	0.6629	2.2361	0.9697
0.75	1.25	0.6392	2.5	0.9787
1.0	1.4142	0.6320	2.8284	0.9816

Owing to the low probability content of the dRMS error circle – today, 95% are generally required for position-finding errors (IMO 1995) - the following quantity is often used:

$$2\,dRMS := 2 * \sqrt{s_x^2 + s_y^2} \qquad \textbf{(5)}$$

2.3 Two-dimensional error probability

The probability that a position fix is within a specific error contour results from the numerical integration of the two-dimensional error distribution (Eq. 6) across the selected error contour, e.g. across a circle or an ellipse.

$$p(a,b) := \frac{1}{2\pi\sigma_x\sigma_y} \iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} e^{\frac{-1}{2}\left(\frac{x^2}{\sigma_x{}^2}+\frac{y^2}{\sigma_y{}^2}\right)} dxdy \qquad \textbf{(6)}$$

The error probabilities of the radial errors dRMS and 2dRMS shown in the table above, i.e. the probability that a position fix lies within a circle with the radius of dRMS or 2dRMS, are available in the referenced literature. They have been recalculated by the author. The algorithm to solve the elliptical integral (Eq. 6) is available in the Web (Harre 2001) and can be used for other error contours, e.g. error ellipses, as well.

2.4 Error ellipses as accuracy contours

Normally, the x and y standard deviations have a different magnitude. The resulting 'natural' *error contour* is an ellipse with the standard deviations as semi-axes. Solutions of the probability integral for specific multiples (*k*) of the standard deviations are shown in the following table.

k	p
1	0.3934
2	0.8646
2.449	**0.95**
3	0.9889

Multiplication of σ_x and σ_y with *k* results in the semi-axes *a* and *b* of the ellipse for the selected *integration contour* (Eq. 6). The probability content of the integration ellipse does not depend on its eccentricity. If the position-finding error is to be displayed in the form of an error ellipse with a probability content of 95%, the standard deviations must be multiplied by a factor 2.449.

At this point it should be noted that the standard deviations must be determined in orthogonal axes, i.e. from uncorrelated errors of the position coordinates. Thus, in the case of LORAN-C measurements for example, it is not sufficient to use distances from two non-rectangular base lines as position errors. Two-dimensional measurements are correlated if the measuring error in one axis also affects the error in the other. The procedure of measured value decorrelation is included in the algorithm made available. The result of such a decorrelation is a rotation of the error ellipse.

2.5 Circular Error Probability

Since the error circles dRMS and 2dRMS comprise error probabilities varying with the ratio of the coordinate error standard deviations, a procedure was developed which can be used to calculate error circles providing a fixed probability content (Harter 1960; Burt, et al. 1965) The error circles of the so-called 'Circular Error Probability (CEP)' comply with this criterion. The procedure is based on tabulated values that are calculated for specified probabilities and graduated ratios of the error standard deviations by evaluating the probability integral. The smaller of the two standard deviations is to be multiplied by a value taken from the table. The result of the calculation is the error circle radius with the desired probability content. Since the use of tables is not optimal for automated calculation, the author has determined approximation polynomials for the tabulated values (Harre 1987). Today, mainly the error circle CEP_{95} containing 95% of the position fixes of a set of measurements is of importance for navigation purposes, the calculation of which is included in the proposed algorithm.

3 Convergence of the systematic position error by position averaging

In this calculation, the values of the position coordinates are initially averaged over the entire data set. The resulting mean values form the coordinates of the Probable Position (PP). Then, the coordinates are averaged with increasing data set number. These 'advancing' mean values and the coordinates of the Probable Position are used to determine the magnitude of a 'progressive' error vector, the '*systematic*' position error, which is displayed vs. elapsed time (Fig. 3, Fig. 4). The respective diagrams show the position deviation from PP as a function of averaging time. With this algorithm the following should be taken into account:

- the overall data set should be large enough so that PP sufficiently approximates the true position to serve as a reference. A reasonable data set would comprise data collected over at least 24 hours, a duration in which the earth has performed a full rotation underneath the satellite constellation,
- second, if PP is used as a reference (for the true position), then the residual error converges against zero, a result which is inherent in the process, but might not entirely reflect reality. An accurately surveyed 'true' position would be the ideal reference to replace PP - if it were available,
- short-term position averaging results taken over an equal time interval vary considerably due to changing Horizontal Dilution of Precision (HDOP) over time.

Nevertheless, the calculation permits an initial assessment of the effect of position averaging and the needed duration to reach a predetermined error level. The error convergence diagrams show that:

- there is little practical benefit for averaging times shorter than 2,000 seconds (Fig. 3, Fig. 4), the same applies for averaging times longer 20,000 seconds (Fig. 5, Fig. 6).
- the averaging effect follows the expected $1/\sqrt{n}$ rule, i.e. if the number of averaged positions (n) is increased, for example, by the factor of 4, the error will be reduced to ½ (Fig. 5, Fig. 6).

4 GPS measurements with and without Selective Availability

The measurements displayed hereafter in the form of graphs were made in Bremen in August 1997 (Fig. 1, 3, 5) and in February 2001 (Fig. 2, 4, 6) using a reference station by Trimble of the 'Pathfinder Community Base Station' (PFCBS) type which was operated as a normal 12-channel GPS receiver. At the beginning of the measurements, the relevant antenna locations were not known. The distance between the two locations is approximately 8 m. Apart from a small sector in the east (shadowing by a tree), the sites permitted a clear view up to low satellite elevations.

In both cases, measurements were evaluated covering a measuring period of approximately 24 hours. The scatter plots with the error contours (Fig. 1, Fig. 2) show the considerable differences in the measuring accuracy of GPS with and without Selective Availability.

4.1 Error levels

Error quantity	with SA	without SA
ó$_x$. ó$_y$ (correlated)	15.68 m . 23.01 m	2.44 m, 2.35 m
dRMS	27.85 m (64%)	3.39 m (63%)
2dRMS	55.69 m (98%)	6.78 m (98%)
CEP_{95}	49.13 m	6.01 m
ó$_x$. ó$_y$ (decorrelated)	15.64 m . 23.04 m	2.84 m , 1.85 m
Azimuth Ell. axis b	3.92°	317.50°

After the discontinuation of SA the random position error, as expressed by CEP_{95}, has dropped to 1/8 or 12.5 % of the previous value.
Measurements performed shortly after the discontinuation of SA show a 95%/24h position error twice as large as the one determined here (JONAS 2000). The author of the referenced paper, however, has confirmed that he even received slightly better results than those mentioned here in measurements performed in March 2001, just before this paper was finalized.

4.2 Scatter plots and error contours

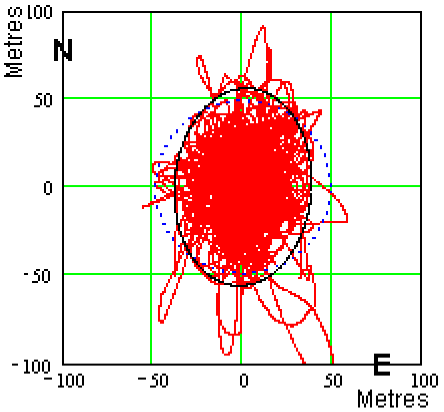

Fig. 1: GPS with SA;
CEP and error ellipse for p=0.95

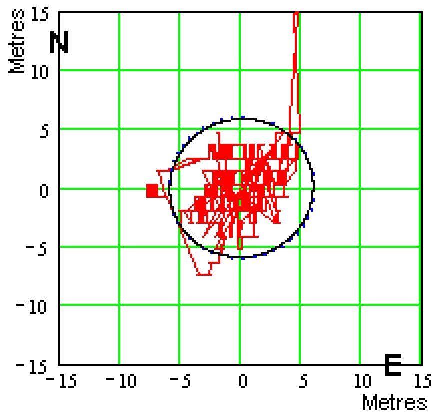

Fig. 2: GPS without SA;
CEP and error ellipse for p=0.95

4.3 Error convergence plots

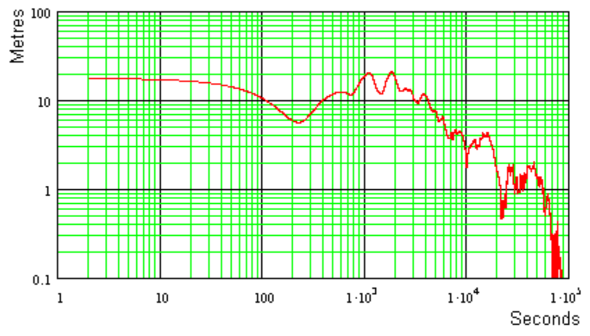

Fig. 3: GPS with SA - the effect of position averaging, logarithmic scales

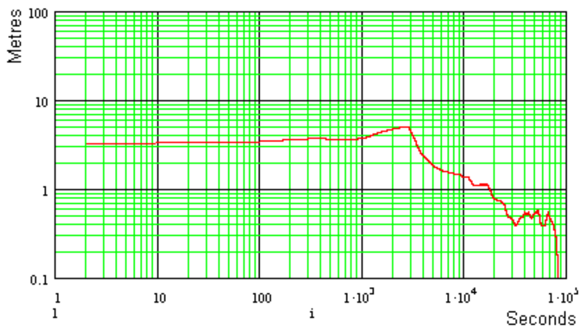

Fig. 4: GPS without SA - the effect of position averaging, logarithmic scales

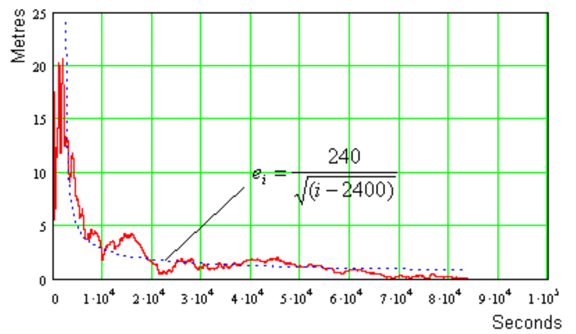

Fig. 5: GPS with SA - the effect of position averaging, linear scales

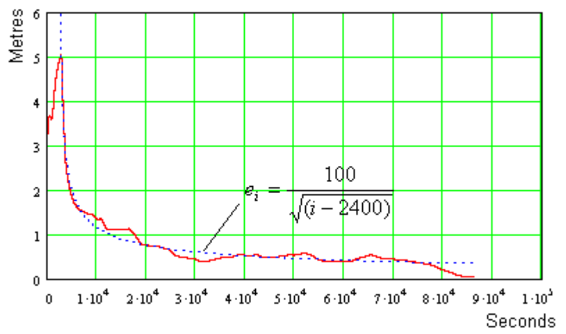

Fig. 6: GPS without SA - the effect of position averaging, linear scales

The comparison of the approximation curves in Fig. 5 and Fig. 6 show that within the data sets examined the 'systematical' position error has dropped to approx ¼or 25% after discontinuation of SA The graphs also show that any given accuracy level is reached much faster than before. While, in the case of SA, a position offset of e.g. 1 meter between the averaged and the reference position is permanently reached only after 60,000 seconds, this result is now available after 12,400 seconds. It is remarkable that averaging has to be continued for approximately 2,000 – 3,000 seconds before improvements with regard to the systematical error become effective (Fig. 3, Fig. 4).

5 Conclusion

Similar to all measurements, position values show statistical behaviour, i.e. two measurements made one after the other normally differ. The position-finding error on the earth's surface can be described by means of two-dimensional statistics. Today, technical literature still includes the error circle measures dRMS and 2dRMS which are easy to calculate but not precise error measures since they do not contain fixed error probabilities.

During the past two decades, much has changed with regard to position finding and navigation. Satellite navigation is unrivaled for its world-wide availability and accuracy. It seems that an increased accuracy also results in higher demand for dependability of position finding. When an average accuracy of perhaps 2 nautical miles has been achieved with the sextant, an error probability of may be 50 % was accepted. Since the introduction of GPS with initial accuracy of better than one hundred meters, a probability of 95 % is demanded for errors specified and obtained in practice. These increasing demands should also be taken into consideration for the applied error measure. Today, an indication without fixed probability is no longer adequate, but is found frequently in the literature. The error quantities to be criticized in this respect were noted as 'confusing' quite some time ago (Bowditch 1977). A general convention on error levels and measures would be sensible.

Very often, electronic chart display and information systems such as ECDIS are used in modern position-finding, navigation and traffic monitoring systems. With today's technology, it should be no problem to visualize the user's position-finding accuracy, upon operator request, by a CEP_{95} error circle or – better still – by a 95% error ellipse. This can enhance safety in special situations during navigation and increase the efficiency in surveying applications.

The GPS measurements evaluated by means of the developed algorithms show an improvement in the random position error by a factor 8 subsequent to discontinuation of Selective Availability. Also, the systematical position error converges much faster than before, if positions are averaged.

The algorithms discussed in this paper are available via the author's web site, in the Internet, in the form of MATHCAD files, and the author hopes that this will trigger further work and publications on questions of navigational accuracy.

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